Exercise 1 (**Optimal Gaussian Approximation of a Rectangular Pulse**). Consider the following two functions defined on $\mathbb{R}$, each normalised to have unit area.

(i) A **Gaussian** centred at the origin with standard deviation $\sigma > 0$:

$$g(x) \;=\; \frac{1}{\sigma\sqrt{2\pi}}\,\exp\!\Big(-\frac{x^2}{2\sigma^2}\Big).$$

(ii) A **rectangular pulse** of width $w > 0$:

$$r(x) \;=\; \begin{cases} \dfrac{1}{w}, & |x| \le \dfrac{w}{2}, \\ 0, & |x| > \dfrac{w}{2}. \end{cases}$$

Find the value of σ (as a function of w) that minimises the squared L^2-distance

$$D(\sigma) \;=\; \int_{-\infty}^{\infty} \big[g(x) - r(x)\big]^2\,dx.$$

Solution. Step 1: Expand the square

Expanding the integrand gives three terms:

$$D(\sigma) \;=\; \underbrace{\int_{-\infty}^{\infty} g^2(x)\,dx}_{I_1} \;-\; 2\underbrace{\int_{-\infty}^{\infty} g(x)\,r(x)\,dx}_{I_2} \;+\; \underbrace{\int_{-\infty}^{\infty} r^2(x)\,dx}_{I_3}.$$

We evaluate each integral in turn.

Step 2: Evaluate I_1

$$I_1 = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2}\,e^{-x^2/\sigma^2}\,dx.$$

Substitute $u = x/\sigma$, so $dx = \sigma\,du$:

$$I_1 = \frac{1}{2\pi\sigma^2}\cdot\sigma\int_{-\infty}^{\infty} e^{-u^2}\,du = \frac{1}{2\pi\sigma}\cdot\sqrt{\pi} = \boxed{\frac{1}{2\sigma\sqrt{\pi}}}.$$

Step 3: Evaluate I_3

Since $r(x) = 1/w$ on an interval of length w and zero elsewhere,

$$I_3 = \int_{-w/2}^{w/2} \frac{1}{w^2}\,dx = \frac{w}{w^2} = \boxed{\frac{1}{w}}.$$

This is a constant independent of σ; it will not affect the optimisation.

Step 4: Evaluate I_2

Because $r(x) = 1/w$ only on $[-w/2,\, w/2]$,

$$I_2 = \frac{1}{w} \int_{-w/2}^{w/2} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/(2\sigma^2)}\, dx.$$

Substitute $u = x/(\sqrt{2}\,\sigma)$, so that $dx = \sqrt{2}\,\sigma\, du$, and define

$$\beta \;\equiv\; \frac{w}{2\sqrt{2}\,\sigma}\,,$$

which maps the integration limits to $\pm\beta$. Then

$$I_2 = \frac{1}{w\,\sigma\sqrt{2\pi}} \cdot \sqrt{2}\,\sigma \int_{-\beta}^{\beta} e^{-u^2}\, du = \frac{1}{w\sqrt{\pi}} \int_{-\beta}^{\beta} e^{-u^2}\, du.$$

Recalling $\displaystyle\int_{-\beta}^{\beta} e^{-u^2}\, du = \sqrt{\pi}\, \operatorname{erf}(\beta)$, we obtain

$$\boxed{I_2 = \frac{\operatorname{erf}(\beta)}{w}}\,.$$

Step 5: Assemble $D(\sigma)$

Combining the three integrals:

$$D = \frac{1}{2\sigma\sqrt{\pi}} \;-\; \frac{2\, \operatorname{erf}(\beta)}{w} \;+\; \frac{1}{w}\,.$$

Step 6: Differentiate with respect to σ and set to zero

We need two derivatives.

First term.

$$\frac{d}{d\sigma}\left(\frac{1}{2\sigma\sqrt{\pi}}\right) = -\,\frac{1}{2\sigma^2\sqrt{\pi}}\,.$$

Second term. Using the chain rule with $\dfrac{d}{d\beta}\operatorname{erf}(\beta) = \dfrac{2}{\sqrt{\pi}}\, e^{-\beta^2}$ and $\dfrac{d\beta}{d\sigma} = -\dfrac{w}{2\sqrt{2}\,\sigma^2}$:

$$\frac{d}{d\sigma}\left[\frac{-2\, \operatorname{erf}(\beta)}{w}\right] = \frac{-2}{w} \;\cdot\; \frac{2}{\sqrt{\pi}}\, e^{-\beta^2} \;\cdot\; \left(-\frac{w}{2\sqrt{2}\,\sigma^2}\right) = \frac{2\, e^{-\beta^2}}{\sqrt{2\pi}\,\sigma^2} = \frac{\sqrt{2}\, e^{-\beta^2}}{\sigma^2\sqrt{\pi}}\,.$$

Setting $dD/d\sigma = 0$:

$$-\,\frac{1}{2\sigma^2\sqrt{\pi}} \;+\; \frac{\sqrt{2}\, e^{-\beta^2}}{\sigma^2\sqrt{\pi}} \;=\; 0.$$

The common factor $1/(\sigma^2\sqrt{\pi})$ cancels, leaving

$$\sqrt{2}\, e^{-\beta^2} = \frac{1}{2}\,.$$

Step 7: Solve for β

$$e^{-\beta^2} = \frac{1}{2\sqrt{2}} \qquad \Longrightarrow \qquad \beta^2 = \ln\bigl(2\sqrt{2}\bigr) = \ln 2 + \tfrac{1}{2}\ln 2 = \frac{3}{2}\ln 2.$$

Hence

$$\boxed{\beta = \sqrt{\tfrac{3}{2}\ln 2} \;\approx\; 1.0179\,.}$$

Step 8: Express σ in terms of w

Recalling $\beta = w/(2\sqrt{2}\,\sigma)$ and solving for σ:

$$\sigma = \frac{w}{2\sqrt{2}\,\beta} = \frac{w}{2\sqrt{2}\,\sqrt{\tfrac{3}{2}\ln 2}} = \frac{w}{2\sqrt{3\ln 2}}\,.$$

$$\boxed{\sigma \;=\; \frac{w}{2\sqrt{3\ln 2}} \;\approx\; 0.347\,w\,.}$$

Verification that this is a minimum

As $\sigma \to 0^+$, the term $I_1 = 1/(2\sigma\sqrt{\pi}) \to \infty$, so $D \to \infty$. As $\sigma \to \infty$, the Gaussian flattens to zero everywhere, so $I_1 \to 0$ and $I_2 \to 0$, giving $D \to I_3 = 1/w$. Since $D(\sigma)$ is continuous on $(0,\infty)$, diverges at $\sigma = 0$, and approaches a finite constant as $\sigma \to \infty$, the unique interior critical point must be a global minimum. $\square$

Exercise 2 (**Optimal Gaussian Approximation of a Cylindrical Pillbox (2-D)**). Consider the following two functions defined on $\mathbb{R}^2$, each normalised to have unit volume.

(i) An isotropic **2-D Gaussian** centred at the origin with standard deviation $\sigma > 0$:

$$g(\mathbf{x}) \;=\; \frac{1}{2\pi\sigma^2}\,\exp\!\Big(-\frac{|\mathbf{x}|^2}{2\sigma^2}\Big), \qquad \mathbf{x} \in \mathbb{R}^2.$$

(ii) A **cylindrical pillbox** of radius $w > 0$:

$$r(\mathbf{x}) \;=\; \begin{cases} \dfrac{1}{\pi w^2}, & |\mathbf{x}| \le w, \\ 0, & |\mathbf{x}| > w. \end{cases}$$

Find the value of σ (as a function of w) that minimises the squared L^2-distance

$$D(\sigma) \;=\; \iint_{\mathbb{R}^2} \big[g(\mathbf{x}) - r(\mathbf{x})\big]^2 \, d^2\mathbf{x}.$$

Solution. All integrands are radially symmetric, so we work in polar coordinates with $\rho = |\mathbf{x}|$ and use $d^2\mathbf{x} = 2\pi\rho\, d\rho$.

Step 1: Expand the square

$$D(\sigma) \;=\; \underbrace{\iint g^2\, d^2\mathbf{x}}_{I_1} \;-\; 2\underbrace{\iint g\, r\, d^2\mathbf{x}}_{I_2} \;+\; \underbrace{\iint r^2\, d^2\mathbf{x}}_{I_3}.$$

Step 2: Evaluate I_1

$$I_1 = \int_0^\infty \frac{1}{4\pi^2\sigma^4}\, e^{-\rho^2/\sigma^2} \;\cdot 2\pi\rho\, d\rho = \frac{1}{2\pi\sigma^4}\int_0^\infty \rho\, e^{-\rho^2/\sigma^2}\, d\rho.$$

With the substitution $t = \rho^2/\sigma^2$ the integral evaluates to $\sigma^2/2$, giving

$$\boxed{I_1 = \frac{1}{4\pi\sigma^2}}.$$

Step 3: Evaluate I_3

$$I_3 = \int_0^w \frac{1}{\pi^2 w^4} \;\cdot 2\pi\rho\, d\rho = \frac{2}{\pi w^4} \;\cdot\; \frac{w^2}{2} = \boxed{\frac{1}{\pi w^2}}.$$

This is independent of σ.

Step 4: Evaluate I_2

Since $r = 1/(\pi w^2)$ only for $\rho \le w$,

$$I_2 = \frac{1}{\pi w^2} \int_0^w \frac{1}{2\pi\sigma^2}\, e^{-\rho^2/(2\sigma^2)} \;\cdot 2\pi\rho\, d\rho = \frac{1}{\pi w^2 \sigma^2} \int_0^w \rho\, e^{-\rho^2/(2\sigma^2)}\, d\rho.$$

With the substitution $t = \rho^2/(2\sigma^2)$, the integral evaluates to $\sigma^2\big(1 - e^{-w^2/(2\sigma^2)}\big)$, so

$$\boxed{I_2 = \frac{1}{\pi w^2}\Big(1 - e^{-w^2/(2\sigma^2)}\Big)}\,.$$

Step 5: Assemble $D(\sigma)$

$$D = \frac{1}{4\pi\sigma^2} \;-\; \frac{2}{\pi w^2}\Big(1 - e^{-w^2/(2\sigma^2)}\Big) \;+\; \frac{1}{\pi w^2}\,.$$

The constant pieces combine:

$$D = \frac{1}{4\pi\sigma^2} \;+\; \frac{2}{\pi w^2}\, e^{-w^2/(2\sigma^2)} \;-\; \frac{1}{\pi w^2}\,.$$

Step 6: Differentiate with respect to σ and set to zero

Let $\alpha \equiv w^2/(2\sigma^2)$ for brevity.

First term.

$$\frac{d}{d\sigma}\left(\frac{1}{4\pi\sigma^2}\right) = -\,\frac{1}{2\pi\sigma^3}\,.$$

Second term. Using $\dfrac{d}{d\sigma}\, e^{-\alpha} = e^{-\alpha} \cdot \dfrac{w^2}{\sigma^3}$:

$$\frac{d}{d\sigma}\left[\frac{2}{\pi w^2}\, e^{-\alpha}\right] = \frac{2}{\pi w^2} \;\cdot\; \frac{w^2}{\sigma^3}\, e^{-\alpha} = \frac{2\, e^{-\alpha}}{\pi\sigma^3}\,.$$

Setting $dD/d\sigma = 0$:

$$-\,\frac{1}{2\pi\sigma^3} \;+\; \frac{2\, e^{-\alpha}}{\pi\sigma^3} \;=\; 0.$$

The common factor $1/(\pi\sigma^3)$ cancels, leaving

$$2\, e^{-\alpha} = \frac{1}{2}\,.$$

Step 7: Solve for α

$$e^{-\alpha} = \frac{1}{4} = 2^{-2} \qquad\Longrightarrow\qquad \alpha = 2\ln 2.$$

Step 8: Express σ in terms of w

Recalling $\alpha = w^2/(2\sigma^2)$:

$$\frac{w^2}{2\sigma^2} = 2\ln 2 \qquad \Longrightarrow \qquad \sigma^2 = \frac{w^2}{4\ln 2}\,.$$

$$\boxed{\sigma \;=\; \frac{w}{2\sqrt{\ln 2}} \;\approx\; 0.601\,w\,.}$$

Verification that this is a minimum

As $\sigma \to 0^+$, $I_1 = 1/(4\pi\sigma^2) \to \infty$, so $D \to \infty$. As $\sigma \to \infty$, the Gaussian flattens to zero, $I_1 \to 0$ and $I_2 \to 0$, giving $D \to I_3 = 1/(\pi w^2)$. By the same continuity argument as in Exercise 1, the unique critical point is a global minimum. $\square$

Exercise 3 (**Optimal Gaussian Approximation of a Uniform Sphere (3-D)**). Consider the following two functions defined on $\mathbb{R}^3$, each normalised to have unit volume.

(i) An isotropic **3-D Gaussian** centred at the origin with standard deviation $\sigma > 0$:

$$g(\mathbf{x}) \;=\; \frac{1}{(2\pi\sigma^2)^{3/2}}\,\exp\!\Big(-\frac{|\mathbf{x}|^2}{2\sigma^2}\Big), \qquad \mathbf{x} \in \mathbb{R}^3.$$

(ii) A **uniform sphere** of radius $w > 0$:

$$r(\mathbf{x}) \;=\; \begin{cases} \dfrac{3}{4\pi w^3}, & |\mathbf{x}| \le w, \\ 0, & |\mathbf{x}| > w. \end{cases}$$

Find the value of σ (as a function of w) that minimises the squared L^2-distance

$$D(\sigma) \;=\; \iiint_{\mathbb{R}^3} \big[g(\mathbf{x}) - r(\mathbf{x})\big]^2\, d^3\mathbf{x}.$$

Solution. All integrands are spherically symmetric, so we work in spherical coordinates with $\rho = |\mathbf{x}|$ and use $d^3\mathbf{x} = 4\pi\rho^2\, d\rho$.

Step 1: Expand the square

$$D(\sigma) \;=\; \underbrace{\iiint g^2\, d^3\mathbf{x}}_{I_1} \;-\; 2\underbrace{\iiint g\, r\, d^3\mathbf{x}}_{I_2} \;+\; \underbrace{\iiint r^2\, d^3\mathbf{x}}_{I_3}.$$

Step 2: Evaluate I_1

$$I_1 = \int_0^\infty \frac{1}{(2\pi\sigma^2)^3}\, e^{-\rho^2/\sigma^2} \;\cdot 4\pi\rho^2\, d\rho = \frac{4\pi}{8\pi^3\sigma^6}\int_0^\infty \rho^2\, e^{-\rho^2/\sigma^2}\, d\rho.$$

Using the standard result $\displaystyle\int_0^\infty \rho^2\, e^{-\rho^2/\sigma^2}\, d\rho = \frac{\sigma^3\sqrt{\pi}}{4}$:

$$I_1 = \frac{1}{2\pi^2\sigma^6} \;\cdot\; \frac{\sigma^3\sqrt{\pi}}{4} = \boxed{\frac{1}{8\pi^{3/2}\sigma^3}}.$$

Step 3: Evaluate I_3

$$I_3 = \int_0^w \frac{9}{16\pi^2 w^6} \;\cdot 4\pi\rho^2\, d\rho = \frac{9}{4\pi w^6} \;\cdot\; \frac{w^3}{3} = \boxed{\frac{3}{4\pi w^3}}.$$

This is independent of σ.

Step 4: Evaluate I_2

Since $r = 3/(4\pi w^3)$ only for $\rho \le w$,

$$I_2 = \frac{3}{4\pi w^3}\int_0^w \frac{1}{(2\pi\sigma^2)^{3/2}}\, e^{-\rho^2/(2\sigma^2)} \cdot 4\pi\rho^2\, d\rho = \frac{3}{4\pi w^3} \cdot \frac{4\pi}{(2\pi)^{3/2}\sigma^3}\int_0^w \rho^2\, e^{-\rho^2/(2\sigma^2)}\, d\rho.$$

Simplifying the prefactor:

$$I_2 = \frac{3}{w^3(2\pi)^{3/2}\sigma^3}\int_0^w \rho^2\, e^{-\rho^2/(2\sigma^2)}\, d\rho.$$

The integral is evaluated by parts. Writing $I = \int_0^w \rho^2\, e^{-\rho^2/(2\sigma^2)}\, d\rho$ with $u = \rho$ and $dv = \rho\, e^{-\rho^2/(2\sigma^2)} d\rho$, so that $v = -\sigma^2\, e^{-\rho^2/(2\sigma^2)}$:

$$I = \left[-\sigma^2 \rho\, e^{-\rho^2/(2\sigma^2)}\right]_0^w + \sigma^2\int_0^w e^{-\rho^2/(2\sigma^2)}\, d\rho = -\sigma^2 w\, e^{-w^2/(2\sigma^2)} + \sigma^3\sqrt{\tfrac{\pi}{2}}\ \mathrm{erf}\Big(\frac{w}{\sigma\sqrt{2}}\Big).$$

Substituting back and noting that $\dfrac{3\sigma^3\sqrt{\pi/2}}{w^3(2\pi)^{3/2}\sigma^3} = \dfrac{3}{4\pi w^3}\cdot\dfrac{\sqrt{\pi}}{\sqrt{\pi}} = \dfrac{3}{4\pi w^3}$ after simplification, we obtain

$$\boxed{I_2 = \frac{3}{4\pi w^3}\left[\mathrm{erf}\Big(\frac{w}{\sigma\sqrt{2}}\Big) - \sqrt{\frac{2}{\pi}}\,\frac{w}{\sigma}\, e^{-w^2/(2\sigma^2)}\right]}.$$

Step 5: Assemble $D(\sigma)$

Write $\alpha \equiv w^2/(2\sigma^2)$ and $a \equiv w/(\sigma\sqrt{2})$ (so $\alpha = a^2/2$ and $\mathrm{erf}(a)$ is the error-function term). Then

$$D = \frac{1}{8\pi^{3/2}\sigma^3} - \frac{3}{2\pi w^3}\left[\mathrm{erf}(a) - \sqrt{\frac{2}{\pi}}\,\frac{w}{\sigma}\, e^{-\alpha}\right] + \frac{3}{4\pi w^3}\,.$$

Step 6: Differentiate with respect to σ and set to zero

Derivative of I_1.

$$\frac{d}{d\sigma}\left(\frac{1}{8\pi^{3/2}\sigma^3}\right) = -\,\frac{3}{8\pi^{3/2}\sigma^4}\,.$$

Derivative of $-2I_2$. We must differentiate

$$F(\sigma)\ \equiv\ \mathrm{erf}(a) - \sqrt{\frac{2}{\pi}}\,\frac{w}{\sigma}\, e^{-\alpha}\,,$$

where $a = w/(\sigma\sqrt{2})$ and $\alpha = w^2/(2\sigma^2)$. We compute the derivative of each piece of F separately.

Piece 1: $\mathrm{erf}(a)$. Using $\dfrac{d}{da}\,\mathrm{erf}(a) = \dfrac{2}{\sqrt{\pi}}\, e^{-a^2}$ and $\dfrac{da}{d\sigma} = -\dfrac{w}{\sqrt{2}\,\sigma^2}$:

$$\frac{d}{d\sigma}\,\mathrm{erf}(a) = \frac{2}{\sqrt{\pi}}\, e^{-a^2}\cdot\left(-\frac{w}{\sqrt{2}\,\sigma^2}\right) = -\frac{w\sqrt{2}}{\sigma^2\sqrt{\pi}}\, e^{-w^2/(2\sigma^2)}.$$

Piece 2: $-\sqrt{2/\pi}\ (w/\sigma)\ e^{-\alpha}$. This is a product of w/σ and $e^{-\alpha}$, where $\alpha = w^2/(2\sigma^2)$. Applying the product rule:

$$\frac{d}{d\sigma}\left[-\sqrt{\frac{2}{\pi}}\ \frac{w}{\sigma}\ e^{-\alpha}\right] = -\sqrt{\frac{2}{\pi}}\ \left[\left(-\frac{w}{\sigma^2}\right) e^{-\alpha} + \frac{w}{\sigma}\ \cdot\ \frac{w^2}{\sigma^3}\ e^{-\alpha}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\ \frac{w\, e^{-\alpha}}{\sigma^2}\left[-1 + \frac{w^2}{\sigma^2}\right] = \sqrt{\frac{2}{\pi}}\ \frac{w\, e^{-\alpha}}{\sigma^2} - \sqrt{\frac{2}{\pi}}\ \frac{w^3\, e^{-\alpha}}{\sigma^4}\,.$$

Combining. Adding the two pieces:

$$F'(\sigma) = \underbrace{-\frac{w\sqrt{2}}{\sigma^2\sqrt{\pi}}\ e^{-\alpha}}_{\text{from erf}}\ +\ \underbrace{\frac{w\sqrt{2}}{\sigma^2\sqrt{\pi}}\ e^{-\alpha} - \sqrt{\frac{2}{\pi}}\ \frac{w^3}{\sigma^4}\ e^{-\alpha}}_{\text{from product}}\,.$$

The first and second terms cancel exactly, leaving

$$F'(\sigma) = -\sqrt{\frac{2}{\pi}}\ \frac{w^3}{\sigma^4}\ e^{-w^2/(2\sigma^2)}.$$

Therefore

$$\frac{d(-2I_2)}{d\sigma} = -\frac{3}{2\pi w^3}\ \cdot\ \left(-\sqrt{\frac{2}{\pi}}\ \frac{w^3}{\sigma^4}\ e^{-\alpha}\right) = \frac{3}{2\pi w^3}\ \sqrt{\frac{2}{\pi}}\ \frac{w^3}{\sigma^4}\ e^{-\alpha} = \frac{3\sqrt{2}}{\sigma^4\, 2\pi^{3/2}}\ e^{-\alpha}\,.$$

Setting $dD/d\sigma = 0$.

$$-\frac{3}{8\pi^{3/2}\sigma^4}\ +\ \frac{3\sqrt{2}}{\sigma^4\, 2\pi^{3/2}}\ e^{-\alpha}\ =\ 0.$$

The common factor $3/(\sigma^4\, 2\pi^{3/2})$ cancels, giving

$$-\frac{1}{4}\ +\ \sqrt{2}\, e^{-\alpha}\ =\ 0 \qquad \Longrightarrow \qquad \sqrt{2}\, e^{-\alpha} = \frac{1}{4}\,.$$

Step 7: Solve for α

$$e^{-\alpha} = \frac{1}{4\sqrt{2}} = 2^{-5/2} \qquad \Longrightarrow \qquad \alpha = \frac{5}{2}\ln 2.$$

Step 8: Express σ in terms of w

Recalling $\alpha = w^2/(2\sigma^2)$:

$$\frac{w^2}{2\sigma^2} = \frac{5}{2}\ln 2 \qquad \Longrightarrow \qquad \sigma^2 = \frac{w^2}{5\ln 2}\,.$$

$$\boxed{\sigma\ =\ \frac{w}{\sqrt{5\ln 2}}\ \approx\ 0.537\, w\,.}$$

Verification that this is a minimum

As $\sigma \to 0^+$, $I_1 = 1/(8\pi^{3/2}\sigma^3) \to \infty$, so $D \to \infty$. As $\sigma \to \infty$, the Gaussian flattens to zero, $I_1 \to 0$ and $I_2 \to 0$, giving $D \to I_3 = 3/(4\pi w^3)$. The unique critical point is therefore a global minimum. $\square$

Remark (**Pattern across dimensions**). A pattern emerges when the results for all three dimensions are compared. In every case we have the form

$$\sqrt{2}\, e^{-\alpha} = C_d\,,$$

where $\alpha = (\text{radius})^2/(2\sigma^2)$ (with "radius" meaning $w/2$ in one dimension and w in two and three dimensions) and C_d is a constant that depends only on the dimension d. The solutions are:

d	Shape	Condition	α	σ	σ/w
1	rect. pulse	$e^{-\alpha} = 2^{-3/2}$	$\frac{3}{2}\ln 2$	$\dfrac{w}{2\sqrt{3\ln 2}}$	≈ 0.347
2	cylinder	$e^{-\alpha} = 2^{-2}$	$2\ln 2$	$\dfrac{w}{2\sqrt{\ln 2}}$	≈ 0.601
3	sphere	$e^{-\alpha} = 2^{-5/2}$	$\frac{5}{2}\ln 2$	$\dfrac{w}{\sqrt{5\ln 2}}$	≈ 0.537

The exponents on the right-hand side follow the sequence $2^{-3/2}$, $2^{-4/2}$, $2^{-5/2}$, i.e. the power is $-(d+2)/2$.

The structural reason this pattern persists is that the same "miraculous" cancellation occurs in every dimension: when the cross term I_2 is differentiated, the contributions from the boundary of the error-function-like integral and from the prefactor of the exponential cancel exactly, leaving a *pure exponential* in α. The derivative of the Gaussian self-energy I_1 is always a power of σ, so the critical-point equation reduces to an algebraic relation between a power of 2 and $e^{-\alpha}$. This mechanism is independent of the dimension, suggesting that the closed-form pattern $e^{-\alpha} = 2^{-(d+2)/2}$ continues to hold for all $d \geq 1$ when the "rectangular" function is taken to be the indicator of a d-dimensional ball, normalised to unit mass, and the Gaussian is the standard isotropic d-dimensional Gaussian.